The Natural Logarithm

It turns out that because the exponential function is so important in calculus, so is its inverse, called the *natural logarithm.* Let's first review inverse functions.

Example 1

In order to be invertible, the graph of a function must pass the **horizontal line test;** in this case, we say that the function is **one-to-one.** That is, *no* horizontal line can pass through more than one point on the graph. So the function $y = x^2$ is not invertible, as we can see in Figure 1.

To get the graph of an inverse function, you reflect the graph along the line $y = x$ (this is why we switch x and y to solve for the inverse function). So if a horizontal line goes through two points on a graph, when you reflect it, a *vertical* line will pass through two points of the inverse graph. But a function must pass the vertical line test – one input can not have more than one output. So $y = x^2$ is not invertible because when you reflect the graph, it fails the vertical line test.

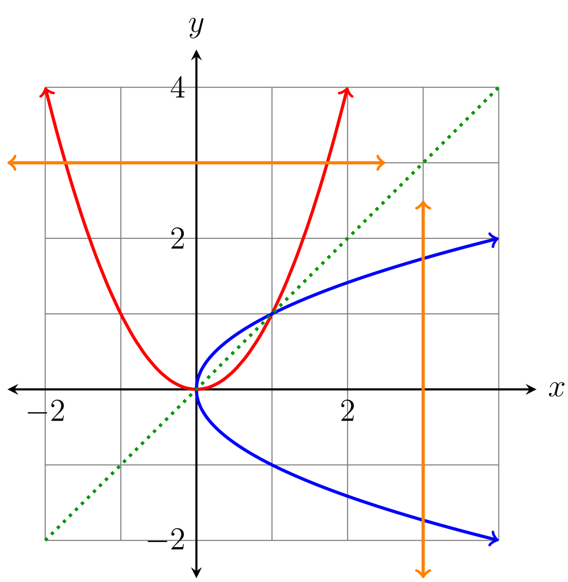

Figure 1: Graph of $y = x^2$ (red) and its reflection (blue) along $y = x$.

In order to create a function which *is* invertible, it is sometimes necessary to restrict the domain. As you can see in Figure 2, if we restrict the domain to $[0, \infty)$, then the graph of $y = x^2$ *does* pass the horizontal line test, and so we can take its inverse.

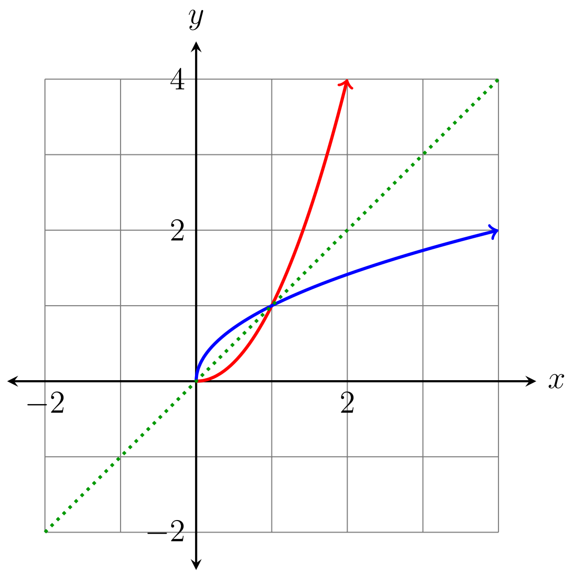

Figure 2: Graph of $y = x^2$ with restricted domain (red) and its reflection (blue) along $y = x$.

This is the *geometry* of inverse functions. What about the algebra of inverse functions? If you have an equation of an invertible function, you just switch x and y (which is the algebraic way to reflect along the line $y = x$) and solve for y. Remember that because we restricted the domain, both x and y are positive, so there is no problem taking square roots.

$$\begin{aligned} y &= x^2 & & \\ x &= y^2 & & \text{switch } x \text{ and } y \\ \sqrt{x} &= y & & \text{take square roots} \\ y &= \sqrt{x} & & \end{aligned}$$

This means that the function $y = \sqrt{x}$ is the inverse function of $y = x^2$ (with restricted domain).

This example is a review of how to find an inverse function. If you feel like you need to review a bit more, see Section 5.2 on p. 378 of the precalculus text on the website.

The Natural Logarithm

Now let's look at taking the inverse function of $y = e^x$. Note that $y = e^x$ is one-to-one, and so we don't need to worry about restricting the domain.

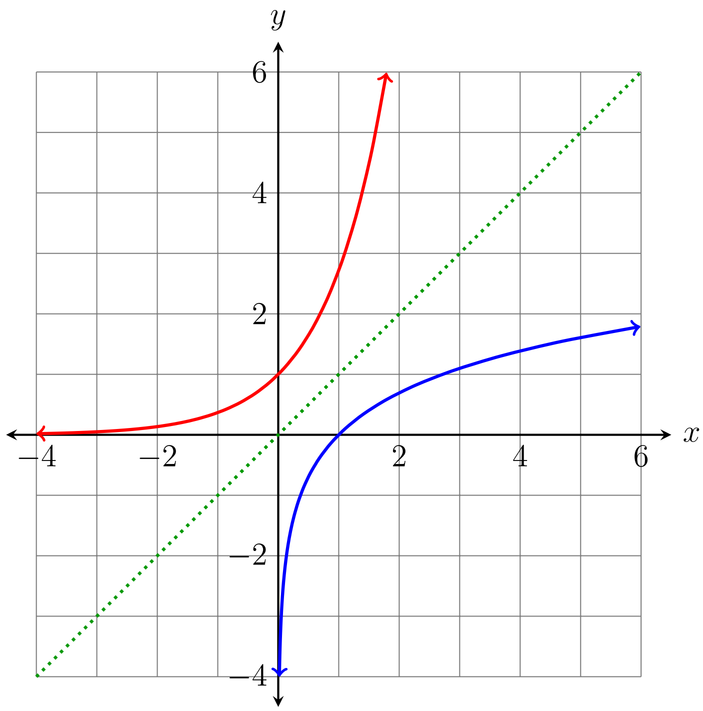

Figure 3: Graph of $y = e^x$ (red) and its reflection (blue) along $y = x$.

Graphically, reflecting $y = e^x$ along $y = x$ isn't difficult. It's the algebra which is a bit tricky. Switching x and y gives $x = e^y$, but the problem is that there is no way to solve for y using algebra that we already know.

The first step is giving a name to this inverse function – it's called the *natural logarithm,* and the notation is $y = \ln x$. There is no *formula* for $\ln x$, so any properties of the natural logarithm have to be deduced from properties of exponential functions.

Since $y = e^x$ and $y = \ln x$ are inverse functions, then

$$\boxed{e^{\ln x} = x, \quad \ln(e^x) = x.}$$

This is just like saying that for $y = x^2$ and $y = \sqrt{x}$, whenever $x \geq 0$, we have

$$(\sqrt{x})^2 = x, \quad \sqrt{x^2} = x.$$

What else can we say about the natural logarithm? Let's figure out a useful property of logarithms. If $a, b > 0$, what can we say about $\ln(ab)$? Put $c = \ln(ab)$ and follow along.

$$\begin{aligned}
\ln(ab) &= c && \\
e^{\ln(ab)} &= e^c && \text{substitute into } e^x \\
ab &= e^x && \text{inverse function property} \\
e^{\ln a} \cdot e^{\ln b} &= e^c && \text{inverse function property} \\
e^{\ln a + \ln b} &= e^c && \text{rules of exponents} \\
\ln a + \ln b &= c && e^x \text{ is one-to-one} \\
\ln(ab) &= \ln a + \ln b. &&
\end{aligned}$$

The important point here is that we used a rule of exponents to get a rule of logarithms by using the fact that exponential functions and logarithms are inverses of each other.

We won't go through deriving all the properties of natural logarithms, but instead summarize them below.

This property	is valid when...
$e^{\ln a} = a$	$a > 0$
$\ln(e^a) = a$	any a
$\ln(ab) = \ln a + \ln b$	$a, b > 0$
$\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$	$a, b > 0$
$\ln(a^m) = m \ln a$	$a > 0$, any m

We remark that $\ln a$ is sometimes called the **logarithm to the base e of a**, and is written $\ln a = \log_e a$. We will look at other bases later; for now, our focus is on the natural logarithm.

Derivative of $\ln x$.

How can we find the derivative of $\ln x$? Using the limit definition is messy – instead we'll use the fact that $\frac{d}{dx}e^x = e^x$ and use the geometry of inverse functions.

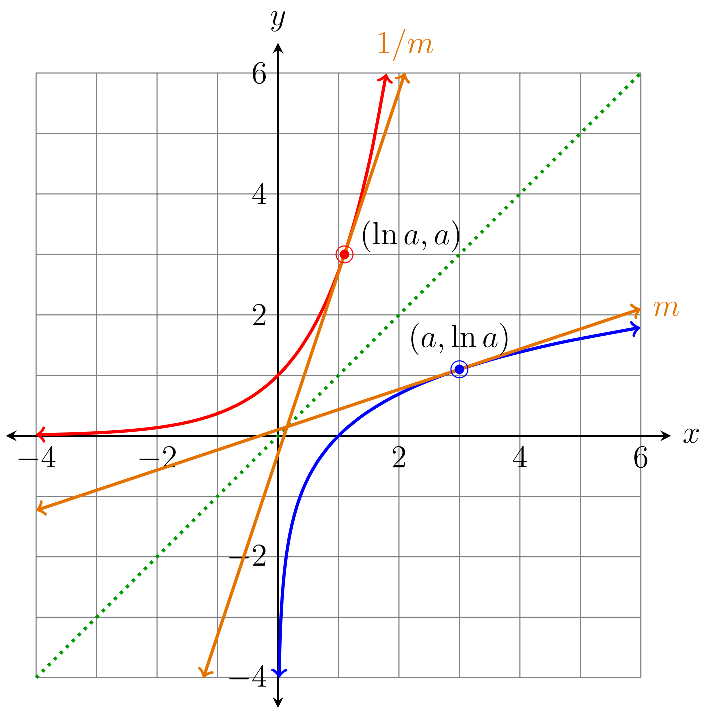

Figure 4: Graph of $y = e^x$ (red) and its reflection (blue) along $y = x$.

It looks like there is a *lot* going on in Figure 4, so let's look it one piece at a time. We'll start with $x = a$, so that $(a, \ln a)$ is on the graph of $y = \ln x$. You can also see the tangent line at this point.

Now let's reflect across the line $y = x$. Algebraically, this amounts to switching x and y values, so now the point $(\ln a, a)$ is on the graph of $y = e^x$. The tangent line here is also drawn.

What happens when we reflect tangent lines? Suppose that you start with a line with slope m (such as the tangent to $y = \ln x$). Then

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}.$$

Switching x and y (that is, reflecting the line) gives

$$\frac{\text{change in } x}{\text{change in } y} = \frac{\text{run}}{\text{rise}} = \frac{1}{m}.$$

In other words, when you reflect a line with slope m over $y = x$, the reflected line has the *reciprocal* slope, $\frac{1}{m}$. (Don't confuse this with perpendicular lines, whose slopes are *negative* reciprocals. There is no negative sign here.)

Up to this point, we've just studied the geometry of Figure 4. Now it's time to use the fact that $\frac{d}{dx}e^x = e^x$. The point $(\ln a, a)$ is on the graph of $y = e^x$. To find the slope of the tangent line, we plug $x = \ln a$ into the derivative of e^x, which is just e^x. Therefore, using a property of inverse functions, the slope of the tangent line to $y = e^x$ Is

$$e^{\ln a} = a.$$

But this means that

$$\frac{1}{m} = a,$$

so that

$$m = \frac{1}{a}.$$

So the slope of the tangent line to $y = \ln x$ at $x = a$ is just $\frac{1}{a}$. But the slope of the tangent line is just the derivative, so we have shown that

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Seems like a lot of work to find a derivative! Finding the inverse function of $y = x^2$ was easy because when we switched x and y to get $x = y^2$, it was easy to solve for y. But for the inverse function of $y = e^x$, there is *no* way to solve $x = e^y$ for y. So we needed to rely heavily on the geometry of inverse functions in order to find the derivative of $y = \ln x$.

Example 2

Suppose $h(x) = \ln(x^3)$. Find $h'(x)$. We can use the Chain Rule here, with

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x},$$

$$g(x) = x^3, \quad g'(x) = 3x^2.$$

Then

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)} \cdot 3x^2 \\ &= \frac{1}{x^3} \cdot 3x^2 \\ &= \frac{3}{x} \end{aligned}$$

It turns out that there is another way to solve this problem. Using a rule of logarithms, we can write

$$h(x) = 3\ln(x).$$

Then we just use the derivative of the logarithm to get $h'(x) = \frac{3}{x}$. This method is simpler, but it does require understanding the rules of logarithms.

Example 3

Suppose $h(x) = \ln(xe^x)$. Find $h'(x)$. Again, let's try the Chain Rule first, with

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x},$$

$$g(x) = xe^x, \quad g'(x) = xe^x + e^x,$$

where $g'(x)$ was found using the Product Rule.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)} \cdot (xe^x + e^x) \\ &= \frac{1}{xe^x} \cdot (xe^x + e^x) \\ &= \frac{xe^x}{xe^x} + \frac{e^x}{xe^x} \\ &= 1 + \frac{1}{x} \end{aligned}$$

Now we'll use the rules of logarithms to find a simpler way. Using two of the rules of logarithms, we can write

$$\begin{aligned} h(x) &= \ln(xe^x) \\ &= \ln x + \ln(e^x) \\ &= \ln x + x \\ h'(x) &= \frac{1}{x} + 1. \end{aligned}$$

It is important to note that simplifying using rules of logarithms is not always possible. But when you can apply the rules, very often the process of taking the derivative is much simpler.

Homework

1. Simplify $\ln(e^6)$.
2. If $a = e^b$, then $b =$ ______________.
3. What is $\ln 1$?
4. Explain, in your own words, why, when you reflect a given line across $y = x$, the slope of the reflected line is the reciprocal of the slope of the given line.
5. Find the derivative of $h(x) = \ln\left(\frac{e^x}{x}\right)$ by (1) using the Chain Rule, (2) using rules of logarithms first to simplify.
6. If $h(x) = \ln(\ln(x))$, find $h'(x)$.
7. Find the equation of the tangent line to $y = \ln x$ at $x = 3$. Check that your answer makes sense numerically by looking at Figure 4.
8. We see from the graph that $y = \ln x$ is increasing. Show this using calculus.
9. We see from the graph that $y = \ln x$ is concave down. Show this using calculus.

Solutions

1. 6, since exponential and logarithmic functions are inverses of each other.
2. $\ln a$, since exponential and logarithmic functions are inverses of each other. We often describe this by saying that "a logarithm is an exponent."
3. 0, since $e^0 = 1$.
4. Answers will be different for everyone.
5. Suppose $h(x) = \ln\left(\frac{e^x}{x}\right)$. First, we'll use the chain rule ($g'(x)$ was found using the Quotient Rule):

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x}$$
$$g(x) = \frac{e^x}{x}, \quad g'(x) = \frac{xe^x - e^x}{x^2}.$$

Then

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)}\left(\frac{xe^x - e^x}{x^2}\right) \\ &= \frac{x}{e^x}\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) \\ &= \frac{x}{e^x}\cdot\frac{e^x}{x} - \frac{x}{e^x}\cdot\frac{e^x}{x^2} \\ &= 1 - \frac{1}{x}. \end{aligned}$$

Next, we'll use rules of logarithms to simplify first.

$$\begin{aligned} h(x) &= \ln\left(\frac{e^x}{x}\right) \\ &= \ln(e^x) - \ln x \\ &= x - \ln x \\ h'(x) &= 1 - \frac{1}{x} \end{aligned}$$

6. Let $h(x) = \ln(\ln x)$. We use the Chain Rule with

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x},$$
$$g(x) = \ln(x), \quad g'(x) = \frac{1}{x}.$$

Then

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)} \cdot \frac{1}{x} \\ &= \frac{1}{x \ln x}. \end{aligned}$$

7. Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$, so $f'(3) = \frac{1}{3}$, which is the slope m of the tangent line. The point $(x_1, y_1) = (3, \ln 3)$ is also on the tangent line, so we have enough information to find an equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \ln 3 &= \frac{1}{3}(x - 3) \\ &= \frac{1}{3}x - 1 \\ y &= \frac{1}{3}x - 1 + \ln 3 \\ &\approx \frac{1}{3}x + 0.1 \end{aligned}$$

Looking at Figure 4, this make sense. You can see that $\frac{\text{rise}}{\text{run}}$ is about $\frac{1}{3}$, and the y-intercept is just slightly above the origin.

8. Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$. But the domain of $f(x)$ is all numbers $x > 0$. Since $x > 0$, then $\frac{1}{x} > 0$ as well, meaning that the function is always increasing.

9. Continuing from the previous problem,

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ &= x^{-1} \\ f''(x) &= -1 \cdot x^{-2} \\ &= -\frac{1}{x^2}. \end{aligned}$$

Since x^2 is always positive, then $f''(x)$ is always negative. This means that the function is concave down.