1 Exponential Functions and e

To begin, review Exponential Functions in the pdf linked to from the website. START on p. 417. STOP on line 6 of p. 418 at the sentence ending "and hence invertible." START at Definition 6 on p. 418. STOP just before Example 6.1.1 on p. 420. This will give you a basic review of exponential functions.

As mentioned in the review, there is a particular choice of base b which we use a *lot* in calculus – the base e. So let's look at why this particular number is important. You want to think of e like π – it is an irrational number which cannot be represented by a fraction. The more you work problems with e, the more you'll get used to it.

You'll need to go to this desmos page on Exponential Derivatives, since the notes here will be referring to the graphs there. You'll see a graph (in red) of $y = b^x$, with a slider you can move to change the base b. Then you also see a graph (in blue) of $f'(x)$.

Right now, we don't have a formula for $f'(x)$. Remember, to use the Power Rule, the x has to be in the *base,* but with $y = b^x$, the x is in the *exponent.* Moreover, b^x cannot be written as product or quotient, so we can't apply these rules, and we can't use the Chain Rule either.

But you should notice one thing. The derivative of an exponential function looks like *another* exponential function. Why should this be?

To help you see this, please go to the desmos page Exponential Tangents. Here, you see the graph of $y = 2^x$. As you move the slider for a, you'll see the tangent line at a along with its slope. (Don't worry about the complicated looking second formula; we won't be needing this.)

So let's take a look at these graphs. As we look at the graph of $y = 2^x$ as we go from -5 to 5, you notice that it starts off very small, gradually increases, and then begins to increasing more rapidly the further right you get. But this is the *same* behavior we notice with the slopes of the tangent lines. The slopes (in blue) start off small, gradually increase, but increase even faster the further we go to the right.

Now a graphical observation is not a mathematical *proof* that the derivative of an exponential function is another exponential, but it is true. These desmos graphs are just meant to show why it makes graphical sense. We will not need a formal proof.

So if the derivative of $f(x) = 2^x$ is another exponential function, just *what* exponential function is it? We won't completely answer this question right now, but at least we'll get a good start on it.

Now navigate back to Exponential Derivatives. Remember, the red graph is $y = b^x$, and the derivative is the blue graph. Now starting at the left of the slider for b, you will notice that the derivative graph is *below* the exponential graph. As you move the slider to the right, you should see that the derivative graph moves up closer to the exponential graph. As b approaches about 2.7, you should notice that the derivative graph crosses over and is now *above* the exponential graph.

What is happening around $b = 2.7$? There is *exactly* one value of b where the derivative graph is *exactly* on top of the exponential graph. For lesser values, the derivative graph is below the exponential graph, and for greater values, the derivative graph is above the exponential graph. The value of b where the two graphs are the same is called e, where $e \approx 2.71828$. The number e is an irrational number; there is no simple formula for it, just like π.

In terms of calculus, we can summarize these observations as follows. Note the important double box!

$$\boxed{\boxed{\frac{d}{dx}e^x = e^x}}$$

This means that the derivative graph of e^x is *exactly* the same graph as e^x. Don't forget this formula! Remember, we need this formula because we can't use the Power Rule – the x is not in the base, it's in the exponent.

Example 1

Let $h(x) = 4e^{0.5x}$. Find $h'(x)$.

Our formula applies *only* if the exponent of e is just x. Since the exponent is different here, we need the Chain Rule. We will use

$$f(x) = e^x, \quad f'(x) = e^x,$$

$$g(x) = 0.5x, \quad g'(x) = 0.5$$

$$\begin{aligned} h(x) &= 4e^{0.5x} \\ h'(x) &= 4f'(g(x))g'(x) \\ &= 4e^{g(x)}(0.5) \\ &= 2e^{0.5x} \end{aligned}$$

Example 2

One application of exponential functions is *bacterial growth.* This is a good model for the initial growth spurt when a culture of bacteria is started in a Petri dish. After a while, though, the Petri dish begins to fill up and the growth rate slows down. To model the slowing part down as well, you'll need to wait until Calculus II.

You'll need to get used to using the variable t for exponential growth, since t is the variable most used for time. So suppose a population of bacteria is modeled by

$$P(t) = 5000e^{0.01t},$$

where P is the population at time t, which is given in hours. Let's look at a few questions.

1. What is the initial population?
2. What is the population after 10 hours?
3. At what rate is the population increasing at 10 hours?

Solutions:

1. The term *initial population* always refers to the time $t = 0$. $P(0) = 5000e^{0.01(0)} = 5000$, so the initial population is 5000 bacteria.
2. After 10 hours, the population is $P(10) = 5000e^{0.01(10)} \approx 5525.85$. (You should have a key on your calculator which calculates e^x.) Since you can't have a fractional number of bacteria, we usually round up and say the population is 5526.
3. Since we're asking for a rate, we need the derivative – just like the velocity is the rate of change of the displacement. We will use the Chain Rule again. We will use the variable t for time.

$$f(t) = e^t, \quad f'(t) = e^t,$$
$$g(t) = 0.01t, \quad g'(t) = 0.01.$$

$$\begin{aligned} P(t) &= 5000e^{0.01t} \\ P'(t) &= 5000f'(g(t))g'(t) \\ &= 5000e^{g(t)}(0.01) \\ &= 50e^{0.01t} \end{aligned}$$

Now use your calculator to see that $P'(10) \approx 55.2585$. So the population increase at 10 hours is approximately 56 bacteria per hour.

Example 3 When calculating derivatives involving e^x, you will almost always need the Chain Rule. Suppose $h(x) = e^{x^2}$. Find $h'(x)$. (Note: e^{x^2} means $e^{(x^2)}$, so you cannot simplify using rules of exponents first. This is just something you have to know; just like the square root sign means a $\frac{1}{2}$ power.)

To do this, we let

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = x^2, \quad g'(x) = 2x.$$

Then

$$\begin{aligned} h(x) &= e^{x^2} \\ h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)}(2x) \\ &= 2xe^{x^2} \end{aligned}$$

Homework

1. Suppose $f(x) = 2xe^x$. Find $f'(x)$.
2. Let $f(x) = \dfrac{e^x}{e^x + 1}$.
3. Let $h(x) = e^x \sin(x)$. Find $h'(x)$.
4. Let $h(x) = e^{\sin(x)+\cos(x)}$. Find $h'(x)$.
5. Suppose $g(x) = e^{\sqrt{x}}$.
6. Suppose a population of bacteria is modeled by

$$P(t) = 4000e^{0.02t},$$

where P is the population at time t, which is given in hours.

   (a) What is the initial population?
   (b) What is the population after 5 hours?
   (c) At what rate is the population increasing at 5 hours?

Solutions:

1. We need the Product Rule here. We'll use

$$f(x) = 2x, \quad f'(x) = 2,$$
$$g(x) = e^x, \quad g'(x) = e^x.$$

Remember the $f(x)$ in the Product Rule is *not* the same as the original $f(x)$. Then

$$\begin{aligned} \frac{d}{dx} 2xe^x &= f(x)g'(x) + g(x)f'(x) \\ &= 2xe^x + e^x(2) \\ &= 2xe^x + 2e^x \\ &= 2e^x(x+1). \end{aligned}$$

It is not necessary to factor out the $2e^x$, but that is likely the answer a book or software would give you.

2. We need the Quotient Rule here. We'll use

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = e^x + 1, \quad g'(x) = e^x.$$

Also, don't forget that

$$e^x \cdot e^x = e^{x+x} = e^{2x}.$$

Then

$$\begin{aligned} \frac{d}{dx} \frac{e^x}{e^x+1} &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{(e^x+1)e^x - e^x \cdot e^x}{(e^x+1)^2} \\ &= \frac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2} \\ &= \frac{e^x}{(e^x+1)^2} \end{aligned}$$

3. We use the Product Rule here.

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = \sin(x), \quad g'(x) = \cos(x).$$

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= e^x(\cos(x)) + \sin(x)(e^x) \\ &= e^x(\cos(x) + \sin(x)) \end{aligned}$$

4. We use the Chain Rule here.

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = \sin(x) + \cos(x), \quad g'(x) = \cos(x) - \sin(x).$$

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)}(\cos(x) - \sin(x)) \\ &= (\cos(x) - \sin(x))e^{\sin(x)+\cos(x)} \end{aligned}$$

5. We use the Chain Rule here.

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = \sqrt{x}, \quad g'(x) = \frac{1}{2\sqrt{x}}.$$

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}e^{\sqrt{x}} \end{aligned}$$

6. (a) The term *initial population* refers to the time $t = 0$. $P(0) = 4000e^{0.02(0)} = 4000$, so the initial population is 4000 bacteria.

   (b) After 5 hours, the population is $P(5) = 4000e^{0.02(5)} \approx 4420.68$. Since you can't have a fractional number of bacteria, we usually round up and say the population is 4421.

   (c) Since we're asking for a rate, we need the derivative – just like the velocity is the rate of change of the displacement. We will use the Chain Rule again.

$$f(t) = e^t, \quad f'(t) = e^t,$$
$$g(t) = 0.02t, \quad g'(t) = 0.02.$$

$$\begin{aligned} P(t) &= 4000e^{0.02t} \\ P'(t) &= 4000f'(g(t))g'(t) \\ &= 4000e^{g(t)}(0.02) \\ &= 80e^{0.02t} \end{aligned}$$

   Now use your calculator to see that $P'(5) \approx 88.4137$. So the population increase at 5 hours is approximately 89 bacteria per hour.