# The Geometry of Second Derivatives

In our discussion of the geometry of the first derivative, we saw that given a function $f(x)$, the graph of the function is increasing at points where $f'(x) > 0$, and decreasing at points where $f'(x) < 0$. But when $f'(x) = 0$, there were three possibilities: a local minimum, a local maximum, or an inflection point. Now if you have a graph, you can just look at it and see which case applies. Here, we will learn how to figure this out using calculus. (You might want to review the graphs in the handout for Day 6 to refresh your memory.)

We will need the **second derivative** here, which is just the derivative *of the derivative.* Let's look at a few examples.

**Example 1**

Suppose $f(x) = x^4 - 3x^2$. Using the power rule, we get

$$f'(x) = 4x^3 - 6x.$$

What if we take the derivative again? We get

$$\begin{aligned} \frac{d}{dx} f'(x) &= \frac{d}{dx}(4x^3 - 6x) \\ &= 12x^2 - 6, \end{aligned}$$

and write

$$f''(x) = 12x^2 - 6.$$

Sometimes you will see the notation

$$\frac{d^2}{dx^2} f(x) = 12x^2 - 6,$$

although we won't be using this notation – $f''(x)$ is much easier to use.

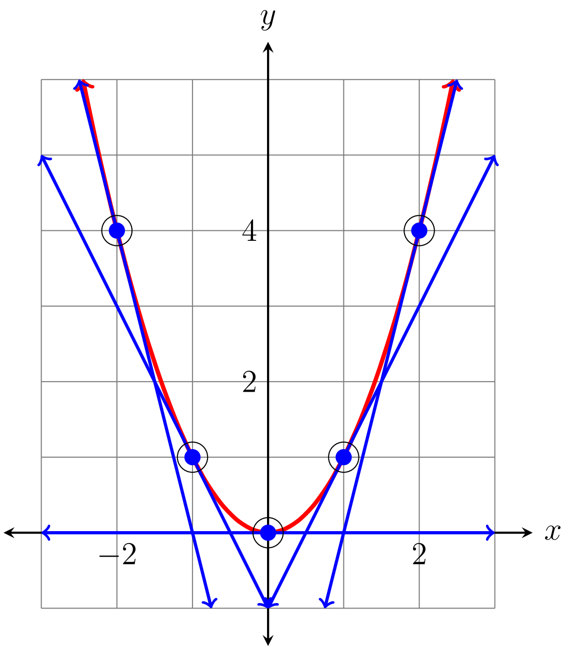
**Example 2**

Suppose $f(x) = \sin(x)$. Then $f'(x) = \cos(x)$, and

$$\begin{aligned} f''(x) &= \frac{d}{dx} \cos(x) \\ &= -\sin(x). \end{aligned}$$

**Example 3**

So finding the second derivative is just a matter of taking the derivative twice in a row. But why would we want to do this? Let's look at an example.




| $x$ | $f'(x)$ | Eqn. of tangent |
|---|---|---|
| $-2$ | $-4$ | $y = -4x - 4$ |
| $-1$ | $-2$ | $y = -2x - 1$ |
| $0$ | $0$ | $y = 0$ |
| $1$ | $2$ | $y = 2x - 4$ |
| $2$ | $4$ | $y = 4x - 1$ |

Figure 1: Graph of $f(x) = x^2$ (left) with table of values of $f'(x)$ (right). For an interactive version of this graph, visit desmos.com.

You can see several tangent lines graphed in Figure 1. In the table on the right, you can see several values of $f'(x)$. What do we notice? That the slopes of the tangent lines – the values of $f'(x)$ – are *increasing* as we go from left to right.

So $f'(x)$ is a function which is increasing. Remember that when a function is increasing, its derivative is positive. Since $f''(x)$ is the derivative of $f'(x)$, this means that

$$\begin{aligned} \frac{d}{dx} f'(x) &> 0, \\ f''(x) &> 0. \end{aligned}$$

Geometrically, we say that when $f''(x) > 0$, the function is **concave up,** meaning essentially, that the function "opens upward." This can happen in three ways, shown in Figure 2.

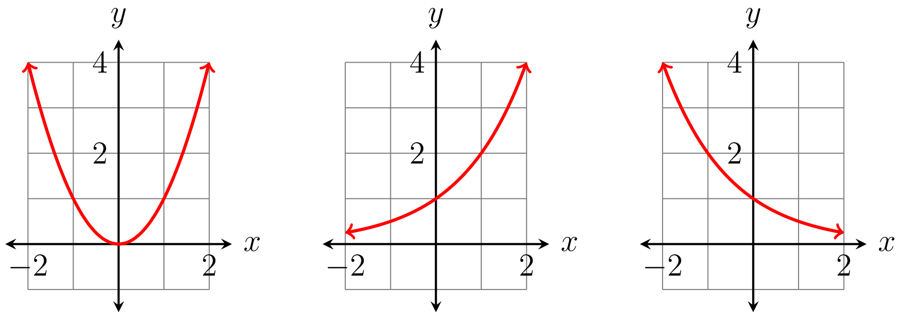



Figure 2: Graph of $f(x) = x^2$ (left), $f(x) = 2^x$ (middle), and $f(x) = 2^{-x}$ (right).

There may be a local minimum involved, as with $f(x) = x^2$ (left in Figure 2). But there may be no minimum at all. The function $f(x) = 2^x$ is always increasing and concave up (middle graph), but $f(x) = 2^{-x}$ is always decreasing and concave up (right graph). So you can see why the first derivative cannot tell us about concavity: a concave up graph could be increasing, decreasing, or both. That's why we need the second derivative.

| |
|---|
| Wherever $f''(x) > 0$, the graph of the function is concave up. |

When $f''(x) < 0$, we say that the function is **concave down.** We won't repeat the previous analysis since it is very similar. Instead, we'll jump to some examples.

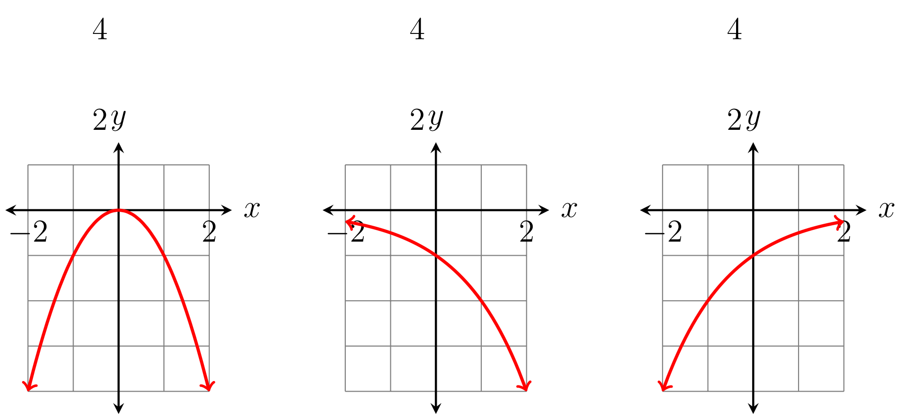



Figure 3: Graph of $f(x) = -x^2$ (left), $f(x) = -2^x$ (middle), and $f(x) = -2^{-x}$ (right).

| |
|---|
| Wherever $f''(x) < 0$, the graph of the function is concave down. |

**Example 4**

The graphs of most functions, though, are partly concave up, and partly concave down. We'll look at such an example, $f(x) = \dfrac{1}{x}$, and apply what we just learned.

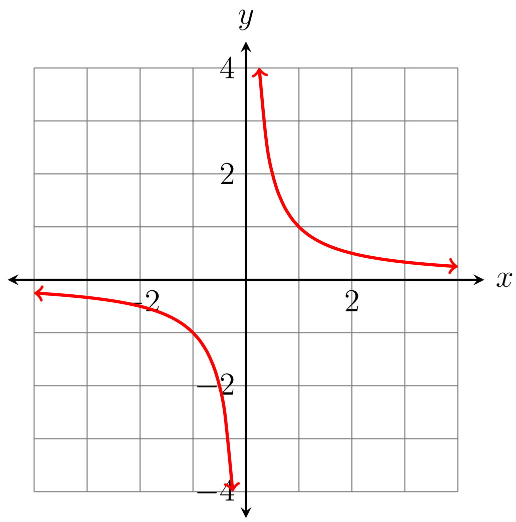



Figure 4: Graph of $f(x) = \dfrac{1}{x}$

Note that $f(x)$ is not defined when $x = 0$. Now let's find $f''(x)$. Remember, we don't need the quotient rule here because we can rewrite the function as $f(x) = x^{-1}$.

$$\begin{aligned} f(x) &= x^{-1} \\ f'(x) &= -x^{-1-1} \\ &= -x^{-2} \\ f''(x) &= -(-2)x^{-2-1} \\ &= 2x^{-3} \end{aligned}$$

So $f''(x) = \dfrac{2}{x^3}$. When $x > 0$, $f''(x) > 0$ since $(+)(+)(+) = (+)$ on the denominator. Hence the graph is concave up on the interval $(0, \infty)$.

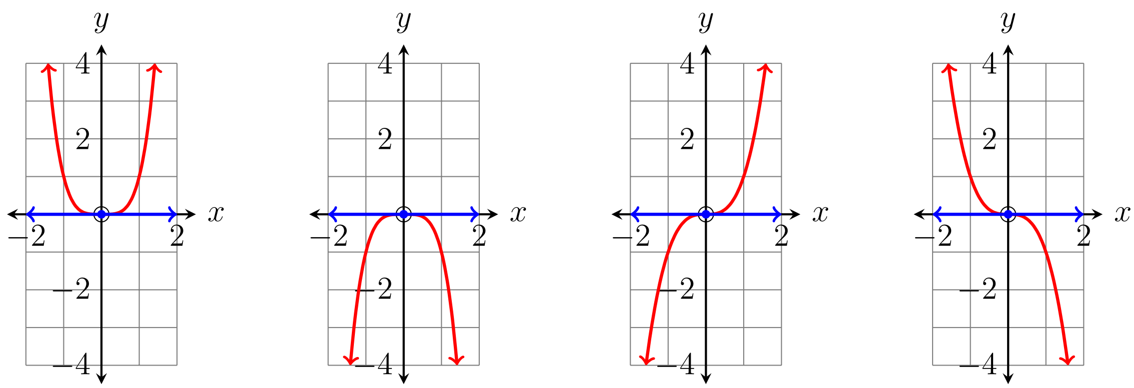
But when $x < 0$, then $f''(x) < 0$ since $(-)(-)(-) = (-)$ on the denominator. Thus the graph is concave down on the interval $(-\infty, 0)$.

Remember that, in general, we use the first derivative, $f'(x)$, to determine where the function is increasing or decreasing. We now know that we use the second derivative, $f''(x)$, to determine where the function is concave up and where it's concave down.

Sometimes it gets confusing to remember what to use – $f(x)$, $f'(x)$, or $f''(x)$ – in a given problem. So here's a summary – so important, it needs to be double-boxed!

| What... | ...it's used for |
|---|---|
| $f(x)$ | Finding $y$-values |
| $f'(x)$ | Increasing/decreasing; minimum/maximum |
| $f''(x)$ | Concave up/down; inflection points |

The only thing left to discuss is the case when $f''(x) = 0$. Like the case when $f'(x) = 0$, there are four possibilities. These are shown in Figure 5.




$f(x) = x^4$ $f(x) = -x^4$ $f(x) = x^3$ $f(x) = -x^3$

Figure 5: Possible behavior when $f''(x) = 0$.

But if we don't have a graph, how can we figure out which is which? We'll use what is called a **sign chart.**

We'll first look at the far left graph in Figure 5, $f(x) = x^4$. To make a sign chart for $f''(x)$ :

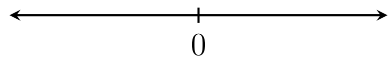
1. Find all values of $x$ where $f''(x) = 0$;
2. Plot these values on a number line;
3. This divides the line into intervals – choose *one* point from each interval (one that is easy to evaluate) and evaluate $f''(x)$; if $f''(x) > 0$, write "+" over the interval, and if $f''(x) < 0$, write "−" above the interval.

So since $f(x) = x^4$, then

$$
\begin{aligned}
f'(x) &= 4x^3 \\
f''(x) &= 4 \cdot 3x^2 \\
&= 12x^2.
\end{aligned}
$$

Let's go through these three steps.

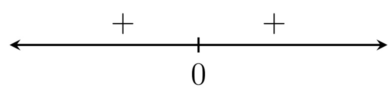
1. If $f''(x) = 12x^2 = 0$, then $x = 0$.
2. This gives the following number line:




3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= 12(-1)^2 \\ &= 12 \\ &> 0 \\ f''(1) &= 12(1^2) \\ &= 12 \\ &> 0. \end{aligned}$$

This yields the following number line:




How do we interpret this number line? Here is an important definition.

> An **inflection point** is a point where a graph *changes* concavity – in other words, it goes from being concave up to concave down, or from concave down to concave up.

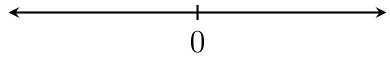
In our example, the graph is concave up on $(-\infty, 0)$ (since $f''(x) > 0$ there), and is *also* concave up on $(0, \infty)$. Thus, the graph does *not* change concavity. Since it is concave up on *both* sides of 0, there is a local minimum at $x = 0$. We can see this by looking at the graph, of course, but a sign chart is necessary when you don't have a graph.

Let's look at another example, this time $f(x) = -x^3$.

1. Since $f(x) = -x^3$, then

$$\begin{aligned} f'(x) &= -3x^2 \\ f''(x) &= -3 \cdot 2x \\ &= -6x. \end{aligned}$$

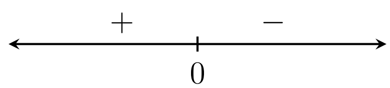
2. This gives the following number line:




3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= -6(-1) \\ &= 6 \\ &> 0 \\ f''(1) &= -6(1) \\ &= -6 \\ &< 0. \end{aligned}$$

This yields the following number line:




Notice that $f(x)$ is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. Since the graph *changes* concavity at $x = 0$, then this is an inflection point. Again, this is clear from the graph – but without a graph, you can still determine whether $x = 0$ is an inflection point or not using a sign graph.

**Example 5**

The sign graphs above were fairly simple. Let's look at a more involved example, using $f(x) = x^4 - 6x^2$.

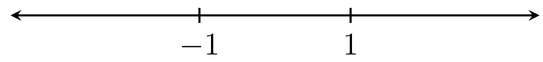
1. Since $f(x) = x^4 - 6x^2$, then

$$\begin{aligned} f'(x) &= 4x^3 - 12x \\ f''(x) &= 12x^2 - 12. \end{aligned}$$

Then

$$\begin{aligned} f''(x) &= 0 \\ 12x^2 - 12 &= 0 \\ 12x^2 &= 12 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

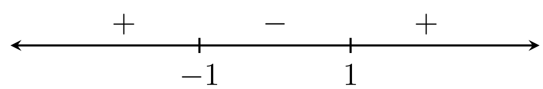
2. This gives the following number line:




3. Now choose one value from each interval. Easy values are $x = -2$, $x = 0$, and $x = 2$.

$$\begin{aligned} f''(-2) &= 12((-2)^2) - 12 \\ &= 36 \\ &> 0 \\ f''(0) &= 12(0^2) - 12 \\ &= -12 \\ &< 0 \\ f''(2) &= 12(2^2) - 12 \\ &= 36 \\ &> 0. \end{aligned}$$

This yields the following number line:




So at $x = -1$, the graph changes from concave up to concave down, and at $x = 1$, the graph changes from concave down to concave up. So $x = -1$ and $x = 1$ are both inflection points.

We can see this on the graph below. Keep in mind that we were able to determine this *without* the graph.

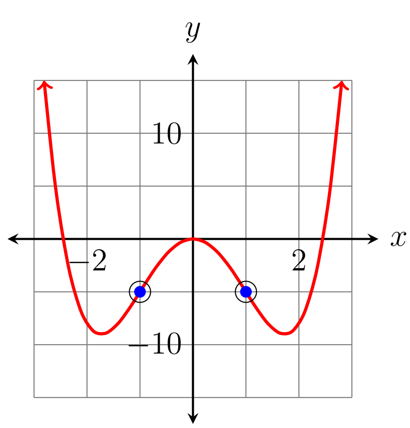



Figure 6: Graph of $f(x) = x^4 - 6x^2$.

## Homework

1. Let $f(x) = x^7 - \dfrac{1}{x}$. Find $f''(x)$.

2. Let $f(x) = \sqrt{x} + \cos(2x)$. Find $f''(x)$.

3. Let $f(x) = \sin^2(x)$. Find $f''(x)$.

4. Fill in the blank with the best answer.

   (a) When $f''(x) < 0$, the graph of $f(x)$ is ____________________.

   (b) We use a sign graph for $f''(x)$ to determine points of ____________________.

   (c) An inflection point is a point on the graph where the ____________________ changes.

5. TRUE FALSE If $f''(x) = 0$, there is an inflection point on the graph at $x$.

6. TRUE FALSE If there is a local maximum or minimum on the graph of $f(x)$, then $f''(x) = 0$ at these points.

7. Let $f(x) = -x^4$. Create a sign graph for $f''(x)$. Follow the same steps as for the graphs in Figure 5.

8. Let $f(x) = \cos(x)$, with the domain being restricted to $[0, 2\pi]$. Using a sign chart, verify that the inflection points are at $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$.

9. Let $f(x) = -x^4 + 24x^2$. Find all inflection points using a sign chart. (Hint: the "24" should make everything come out nicely.)

## Solutions

1.

$$\begin{aligned} f(x) &= x^7 - x^{-1} \\ f'(x) &= 7x^6 + x^{-2} \\ f''(x) &= 42x^5 - 2x^{-3} \end{aligned}$$

2.

$$\begin{aligned} f(x) &= x^{1/2} + \cos(2x) \\ f'(x) &= \frac{1}{2}x^{-1/2} - 2\sin(2x) & \text{Chain rule}: f(x) = \cos(x), g(x) = 2x \\ f''(x) &= -\frac{1}{2}x^{-3/2} - 4\cos(2x) & \text{Chain rule}: f(x) = \sin(x), g(x) = 2x. \end{aligned}$$

3.

$$\begin{aligned} f(x) &= \sin^2(x) \\ f'(x) &= 2\sin(x)\cos(x) & \text{Chain rule}: f(x) = x^2, g(x) = \sin(x) \\ f''(x) &= (2\sin(x))(-\sin(x)) + (\cos(x))(2\cos(x)) & \text{Product rule}: f(x) = 2\sin(x), g(x) = \cos(x) \\ &= 2(\cos^2(x) - \sin^2(x)) \end{aligned}$$

4. (a) concave down.
   (b) inflection.
   (c) concavity.

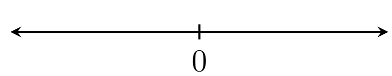
5. FALSE (could be a local minimum or maximum)

6. FALSE (not true when $f(x) = x^2$)

7. Let $f(x) = -x^4$. To make a sign chart for $f''(x)$:

$$\begin{aligned} f'(x) &= -4x^3 \\ f''(x) &= -4 \cdot 3x^2 \\ &= -12x^2. \end{aligned}$$

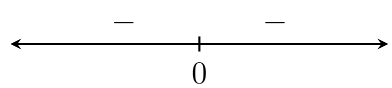
   (a) If $f''(x) = -12x^2 = 0$, then $x = 0$.
   (b) This gives the following number line:



(c) Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= -12(-1)^2 \\ &= -12 \\ &< 0 \\ f''(1) &= -12(1^2) \\ &= -12 \\ &< 0. \end{aligned}$$

This yields the following number line:




Since the concavity does *not* change, there is no inflection point. Since the graph stays concave down, there is a local maximum at $x = 0$.

8. Let $f(x) = \cos(x)$.

(a) Then

$$\begin{aligned} f(x) &= \cos(x) \\ f'(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \end{aligned}$$

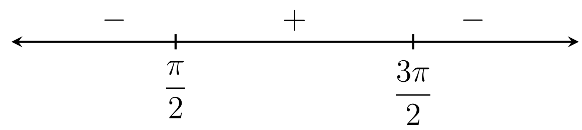
$\cos(x) = 0$ when $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$ on the interval $[0, 2\pi]$; we know this from looking at the unit circle.

(b) This gives the following number line:


(c) Looking at easy values in each interval, we choose $x = \dfrac{\pi}{4}, \pi, \dfrac{7\pi}{4}$. Then

$$\begin{aligned} f''(\pi/4) &= -\cos(\pi/4) \\ &= -1/\sqrt{2} \\ &< 0 \\ f''(\pi) &= -\cos(\pi) \\ &= 1 \\ &> 0 \\ f''(7\pi/4) &= -\cos(7\pi/4) \\ &= -1/\sqrt{2} \\ &< 0 \end{aligned}$$

This yields the following number line:




Since concavity changes at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, these must be inflection points.

9. Let $f(x) = -x^4 + 24x^2$.

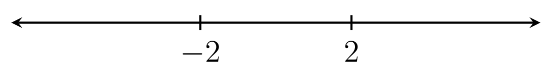
   (a) Since $f(x) = -x^4 + 24x^2$, then

$$\begin{aligned} f'(x) &= -4x^3 + 48x \\ f''(x) &= -12x^2 + 48. \end{aligned}$$

   Then

$$\begin{aligned} f''(x) &= 0 \\ -12x^2 + 48 &= 0 \\ -12x^2 &= -48 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

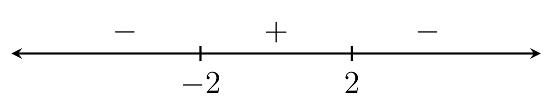
   (b) This gives the following number line:




   (c) Now choose one value from each interval. Easy values are $x = -3$, $x = 0$, and $x = 3$.

$$\begin{aligned} f''(-3) &= -12((-3)^2) + 48 \\ &= -60 \\ &< 0 \\ f''(0) &= -12(0^2) + 48 \\ &= 48 \\ &> 0 \\ f''(3) &= -12(3^2) + 48 \\ &= -60 \\ &< 0. \end{aligned}$$

   This yields the following number line:



So at $x = -2$, the graph changes from concave down to concave up, and at $x = 2$, the graph changes from concave up to concave down. So $x = -2$ and $x = 2$ are both inflection points.