1 # 1 Rules of Differentiation

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Power Rule:	$\dfrac{d}{dx}(x^n) = nx^{n-1}$
Sum Rule:	$\dfrac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
Difference Rule:	$\dfrac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
Constant Multiple Rule:	$\dfrac{d}{dx}(cf(x)) = cf'(x)$
Product Rule:	$\dfrac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
Quotient Rule:	$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$
Chain Rule:	$\dfrac{d}{dx}(f \circ g)(x) = f'(g(x))g'(x)$

3 **Exercises**

1. Review Paul's Online Notes, or other resource, about function composition. In these exercises, we look at *reverse* function composition. As an example, suppose you are given $h(x) = \sqrt{x^2+1}$. Can you find $f(x)$ and $g(x)$ such that $h = f \circ g$? Since $f \circ g(x) = f(g(x))$, g is the *first* thing you do, and f is the *last* thing you do. So we would have $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$. Check:

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(x^2+1) \\ &= \sqrt{x^2+1}. \end{aligned}$$

4 For each of the following functions, find $f(x)$ and $g(x)$ so that $h = f \circ g$.

5 (a) $h(x) = (2x-1)^5$

6 (b) $h(x) = \sin^2(x)$

7 (c) $h(x) = \dfrac{1}{x^3+x}$

8 (d) $h(x) = \cos(2x+1)$

2. Use this step-by-step process to prove the Quotient Rule. Begin by writing

$$y(x) = \frac{f(x)}{g(x)}.$$

(a) Multiply both sides of this equation by $g(x)$. This eliminates the fraction.
(b) Now apply the product rule to $f(x)$.
(c) You should now see both a $y(x)$ and a $y'(x)$ on one side of the equation. Substitute $y(x) = \dfrac{f(x)}{g(x)}$.
(d) Since this gives you a fraction, multiply both sides by $g(x)$.
(e) The fraction should go away now. Solve for $y'(x)$, and you should have the Quotient Rule.

Solutions

1. (a) $f(x) = x^5$, $g(x) = 2x - 1$.
   (b) $f(x) = x^2$, $g(x) = \sin(x)$.
   (c) $f(x) = \dfrac{1}{x}$, $g(x) = x^3 + x$.
   (d) $f(x) = \cos(x)$, $g(x) = 2x + 1$.

2. Let $y(x) = \dfrac{f(x)}{g(x)}$.
   (a) Note how $g(x)$ cancels.

$$y(x) \cdot g(x) = \frac{f(x)}{\cancel{g(x)}} \cdot \cancel{g(x)}$$
$$f(x) = y(x) \cdot g(x)$$

   (b) Now we use the product rule. You have to be careful to use y and g, not f and g.

$$f'(x) = y(x) \cdot g'(x) + g(x) \cdot y'(x)$$

   (c) Now make the substitutione $y(x) = \dfrac{f(x)}{g(x)}$.

$$f'(x) = \frac{f(x)}{g(x)} \cdot g'(x) + g(x) \cdot y'(x)$$

   (d) Multiply both sides by $g(x)$ and simplify.

$$g(x) \cdot f'(x) = \cancel{g(x)} \cdot \frac{f(x)}{\cancel{g(x)}} \cdot g'(x) + g(x) \cdot g(x) \cdot y'(x)$$
$$g(x) \cdot f'(x) = f(x) \cdot g'(x) + (g(x))^2 y'(x)$$

   (e) Now solve for $y'(x)$. Do this by getting the $y'(x)$ term by itself, and then dividing the whole equation by $(g(x))^2$.

$$(g(x))^2 y'(x) = g(x) \cdot f'(x) - f(x) \cdot g'(x)$$
$$y'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$