1 The Derivative of $y = \sin(x)$.

Let's look at the derivatives of a few common functions. We'll start with $f(x) = \sin(x)$; two full periods are graphed in Figure 1. What is the slope of the tangent line at $x = 0$? In other words, what is $f'(0)$?

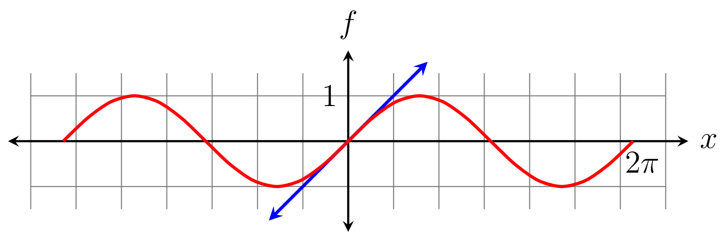

Figure 1: Graph of $f(x) = \sin(x)$ with tangent line at $x = 0$.

We will use the following definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We first start with $x = 0$, write out the definition substituting in 0 for x, and then simplify a little bit.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{\sin(h) - \sin(0)}{h} && \text{since } f(x) = \sin(x) \\ &= \lim_{h \to 0} \frac{\sin(h)}{h} && \text{since } \sin(0) = 0. \end{aligned}$$

Up until now, we were able to use algebra to make the "h" cancel out so we could just substitute $h = 0$. But it is not possible to do that here. So how do we proceed?

There are two other ways we can look at limits: numerically and graphically. We'll start with numerically. Since we are looking at a limit at $h \to 0$, you can use your calculator to look at the quotient $\dfrac{\sin(h)}{h}$ for values of h closer and closer to 0.

I set my calculator to radian mode (important!) and rounded to six decimal places. As h gets closer to 0 from the left and right, it looks like the quotient $\dfrac{\sin(h)}{h}$ gets closer and closer to 1. Using limit notation, we would write

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1.$$

h	$\sin(h)/h$
-0.1	0.998334
-0.01	0.999983
-0.001	1.000000
-0.0001	1.000000
0.1	0.998334
0.01	0.999983
0.001	1.000000
0.0001	1.000000

Table 1: Approximating $\lim_{h\to 0} \dfrac{\sin(h)}{h}$.

It is worth noting that if your calculator were in degree mode, it would look like this limit is approximately 0.017453. Units of radians make trigonometry much easier (as far as calculus is concerned). This is very similar to choosing appropriate units in science. The metric system is far better suited to science than inches, ounces, etc.

Another way to guess $\lim_{h\to 0} \dfrac{\sin(h)}{h}$ is to look at the ratio $\dfrac{\sin(h)}{h}$ as a function itself, as in Figure 2.

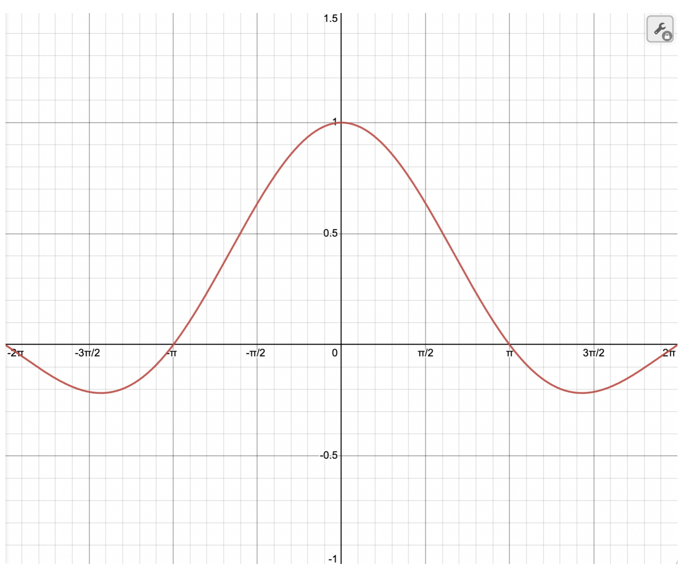

Figure 2: Graph of $g(h) = \dfrac{\sin(h)}{h}$.

I made with graph with desmos and took a screen shot. It looks like it crosses the y-axis at 1 (just like the limit). It's important to say "looks like" since you can't actually evaluate

$$g(0) = \frac{\sin(0)}{0} = \frac{0}{0}.$$

But most graphing programs are able to "fill in the hole" at $x = 0$ to get a smooth curve.

There is also a more complicated mathematical proof using geometry and trigonometry, but it's more than we need. For the functions we'll be looking at, if looking at a limit numerically and graphically gives the same result, then you can be sure you've found the right limit.

To recap, we found that

$$\boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1.}$$

But this was the limit we needed to evaluate to find the slope of the tangent line at $x = 0$, and so the slope of this line is 1.

We started just with looking at $x = 0$ since we needed to see different ways to evaluate limits. So with $f(x) = \sin(x)$, let's find $f'(x)$ for *every* x. To do this, we'll need an identity from trigonometry:

$$\boxed{\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b).}$$

Now let's start with the definition.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} && \text{since } f(x) = \sin(x) \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} && \text{using the identity} \end{aligned}$$

This looks a bit more complicated than other limits we've seen. Let's take a few steps to rearrange terms.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \left[\sin(x)\cdot\frac{\cos(h)}{h} + \cos(x)\cdot\frac{\sin(h)}{h} - \frac{\sin(x)}{h}\right] && \text{splitting apart} \\ &= \lim_{h\to 0} \left[\sin(x)\left(\frac{\cos(h)-1}{h}\right) + \cos(x)\cdot\frac{\sin(h)}{h}\right] && \text{combining } \sin(x) \text{ terms} \end{aligned}$$

Now since h goes to 0, x does not change in this limit. So we can factor out terms *only* containing x. If the limit involves $h \to a$, you can *never* factor out any expression containing h from the limit.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left[\sin(x)\left(\frac{\cos(h)-1}{h}\right)+\cos(x)\cdot\frac{\sin(h)}{h}\right] \\ &= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \end{aligned}$$

The first limit might not look familiar, but the second one does – we started off by finding this exact limit: it is 1.

What about the first limit? Since we already worked one limit like this in detail, we won't do another one. But when you look at this limit numerically and graphically, you see that:

$$\boxed{\lim_{h\to 0}\frac{\cos(h)-1}{h}=0.}$$

How can we use this? Basically, we rewrote the quotient $\dfrac{f(x+h)-f(x)}{h}$ in such a way that is involves limits we can derive numerically and graphically. So we just substitute in the values of these limits.

$$\begin{aligned} f'(x) &= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \\ &= \sin(x)\cdot 0+\cos(x)\cdot 1 \\ &= \cos(x). \end{aligned}$$

Done! So when $f(x) = \sin(x)$, then $f'(x) = \cos(x)$. So $\cos(x)$ is the derivative of $\sin(x)$, which we often write

$$\boxed{\frac{d}{dx}\sin(x)=\cos(x).}$$

Exercises

1. Show numerically and graphically that
$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0.$$

2. By following a similar sequence of steps as for $\sin(x)$, but using a different trigonometric identity, show that
$$\frac{d}{dx}\cos(x) = -\sin(x).$$

3. Let $f(x) = \sqrt{x+1}$, so that $f'(x) = \dfrac{1}{2\sqrt{x+1}}$.
   (a) Graph this function on desmos.
   (b) Where is $f'(x)$ defined?
   (c) What is $f'(3)$?
   (d) Find the equation of the tangent line at $x = 3$.
   (e) Where is $f'(x) = 0$?
   (f) Where is the function increasing?
   (g) Where is the function decreasing?

4. Let $f(x) = x^3 - 3x$.
   (a) Graph this function on desmos.
   (b) Using the definition of the derivative, show that $f'(x) = 3x^2 - 3$.
   (c) Where is $f'(x)$ defined?
   (d) What is $f'(2)$?
   (e) Find the equation of the tangent line at $x = 2$.
   (f) Where is $f'(x) = 0$? Does the function have a local minimum, local maximum, or an inflection point at these points?
   (g) Where is the function increasing?
   (h) Where is the function decreasing?

Solutions

3. (a) Do this online.

   (b) $f'(x)$ is defined when $x+1>0$, since you can't have 0 in the denominator or a negative number inside a square root. So $x>-1$, or $(-1,\infty)$ using interval notation.

   (c)

$$f'(3)=\frac{1}{2\sqrt{3+1}}=\frac{1}{4}.$$

   (d) We know the slope is $\frac{1}{4}$ from (c). $f(3)=\sqrt{3+1}=2$, so a point on the line is $(3,2)$.

$$\begin{aligned}
y-y_1&=m(x-x_1)\\
y-2&=\frac{1}{4}(x-3)\\
y-2&=\frac{1}{4}x-\frac{3}{4}\\
y&=\frac{1}{4}x+\frac{5}{4}
\end{aligned}$$

   (e) $f'(x)$ can never be 0.

   (f) Recall that $\sqrt{x+1}$ can never be negative, and it can't be 0 either (since you can't have a 0 in the denominator), and so it is always positive. Therefore, the function is increasing on $(-1,\infty)$.

   (g) Because the derivative is never negative, the function is never decreasing.

4. (a) Do this online.

   (b)

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{(x+h)^3-3(x+h)-(x^3-3x)}{h}\\
&=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{3x}-3h-\cancel{x^3}+\cancel{3x}}{h}\\
&=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}\\
&=\lim_{h\to 0}\left(\frac{3x^2h}{h}+\frac{3xh^2}{h}+\frac{h^3}{h}-\frac{3h}{h}\right)\\
&=\lim_{h\to 0}(3x^2+3xh+h^2-3)\\
&=3x^2-3.
\end{aligned}$$

(c) Polynomials are defined for *all* x. In interval notation, $f(x)$ is defined on $(-\infty, \infty)$.

(d)

$$f'(2) = 3(2^2) - 3 = 9.$$

(e) We know the slope is 9 from (d). Since $f(2) = 2$, we use the point $(2, 2)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 9(x - 2) \\ y - 2 &= 9x - 18 \\ y &= 9x - 16 \end{aligned}$$

(f)

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 3 &= 0 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Looking at the graph, we observe a local maximum at $x = -1$ and a local minimum at $x = 1$.

(g) The function is increasing whenever $f'(x) > 0$.

$$\begin{aligned} f'(x) &> 0 \\ 3x^2 - 3 &> 0 \\ 3x^2 &> 3 \\ x^2 &> 1 \end{aligned}$$

We have to be careful with $x^2 > 1$, don't just take square roots too quickly. You'll get $x > 1$, which is correct. But you also get $x < -1$. In interval notation, this is $(-\infty, -1) \cup (1, \infty)$, where the symbol "$\cup$" is "union" and means graphing both intervals on the same number line.

(h) The function is decreasing whenever $f'(x) < 0$. So we just have to change "$>$" to "$<$" in the previous work, giving $x^2 < 1$. This happens in the interval $(-1, 1)$.