The Geometry of Derivatives

We just learned how to find a derivative using the geometric definition derived from looking at secant lines. The process of finding a derivative *algebraically* is sometimes rather tedious. Here, we'll look at the *geometrical* meaning of the derivative. Because we want to emphasize the important concepts, we'll look at a basic function, $f(x) = x^2$, shown below, with it's derivative, $f'(x) = 2x$.

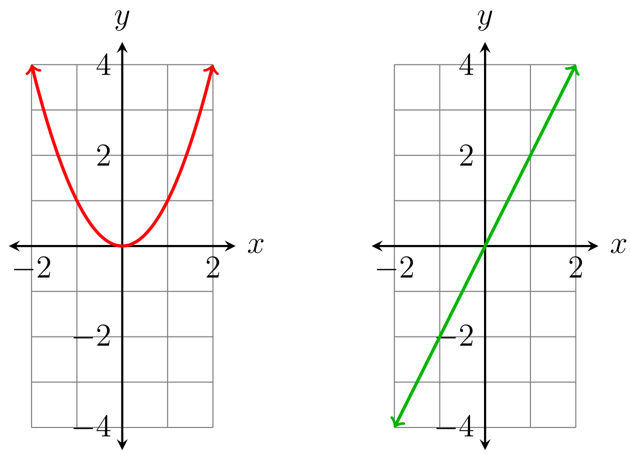


Figure 1: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right).

Let's look at exactly what knowledge we can gain by knowing the derivative. First, we can find the slope of a tangent at any given point. So, since $f(1) = 1$, we know that the tangent line at $(1, 1)$ has a slope of $f'(1) = 2 \cdot 1 = 2$, shown below. Note the corresponding point on the derivative graph, $(1, 2)$.

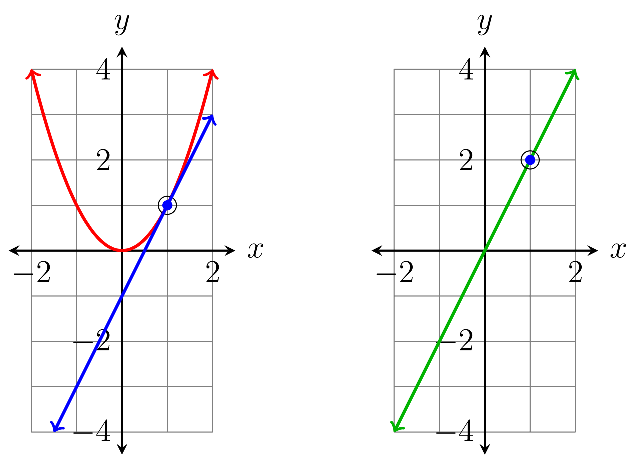


Figure 2: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right).

We now have enough information to work out an equation for the tangent line at $x = 1$, since we know the slope $m = 2$ and a point $(1, 1)$ on the line. For reference, we recall that a line with slope m which passes through the point (x_1, y_1) can be described by the following equation:

$$\boxed{y - y_1 = m(x - x_1).}$$

Substituting in our values: $m = 2$, $x_1 = 1$, and $y_1 = 1$, we get

$$y - 1 = 2(x - 1),$$

which simplifies to $y = 2x - 1$. In Figure 2, you can observe that the slope is 2 and the y-intercept is 1. So we can use the derivative to find an equation of the tangent line at a point.

What else does the derivative tell us? Let's look now at the case when $x > 0$; we look at the specific case $x = 0.5$ in the graphs below.

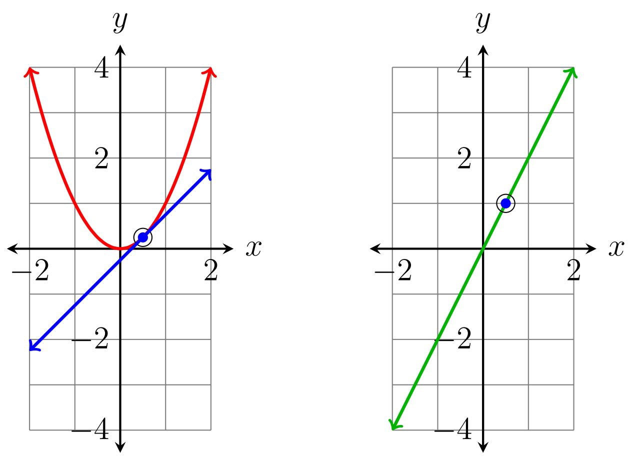


Figure 3: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right); $x = 0.5$.

Notice that the graph is *increasing* when $x > 0$, and so the tangent line has a *positive* slope. We can see this by looking at the graph. But in addition to this, we have, when $x > 0$,

$$\begin{aligned} f'(x) &= 2x \\ &> 2 \cdot 0 && \text{since } x > 0 \\ &= 0. \end{aligned}$$

We can summarize this as follows:

If $f'(x) > 0$ for some value of x, then the function $f(x)$ is increasing at that value of x.

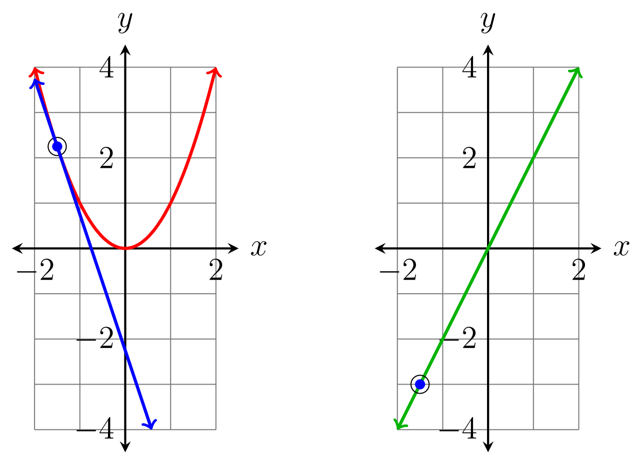


Figure 4: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right); $x = -1.5$.

Now let's look at the case when $x < 0$; the case when $x = -1.5$ is graphed in Figure 4.

In this case, the graph is *decreasing* when $x < 0$, and so the tangent line has a *negative* slope. We can see this by looking at the graph. But we can also see this algebraically; when $x < 0$,

$$\begin{aligned} f'(x) &= 2x \\ &< 2 \cdot 0 && \text{since } x < 0 \\ &= 0. \end{aligned}$$

We can summarize this case as follows:

> If $f'(x) < 0$ for some value of x, then the function $f(x)$ is decreasing at that value of x.

Finally, let's look at what happens when $f'(x) = 0$. Since $f'(x) = 2x$, it follows that $f'(x) = 0$ when $x = 0$. This is illustrated below.

Notice that because we have a minimum at $x = 0$, the tangent line there is horizontal (see Figure 5).

Now let's put all these things together.

$f'(x)$	Where	What happens
$f'(x) < 0$	$(-\infty, 0)$	$f(x)$ is decreasing
$f'(x) = 0$	$x = 0$	minimum
$f'(x) > 0$	$(0, \infty)$	$f(x)$ is increasing

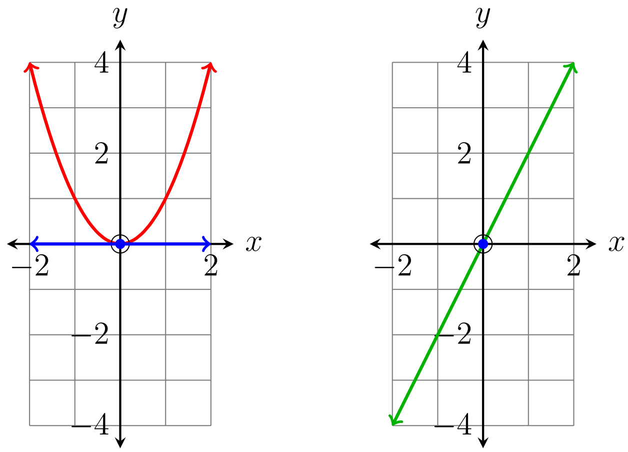
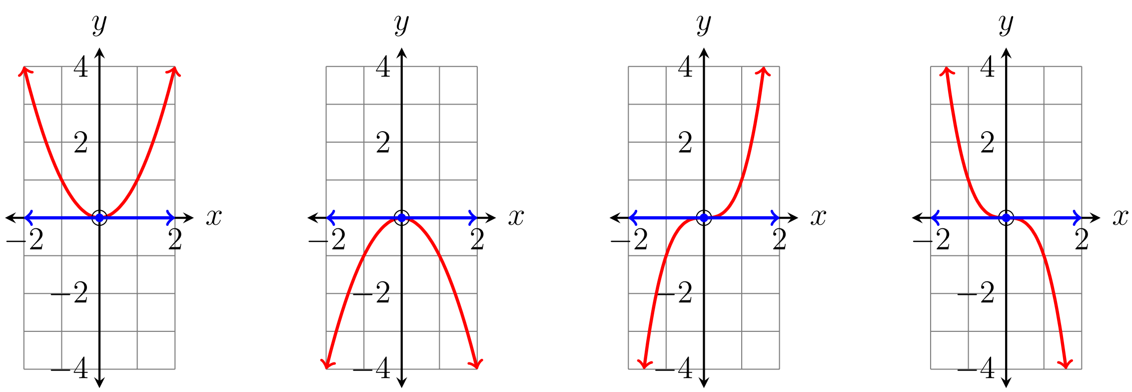


Figure 5: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right); $x = 0$.

So we can tell a lot about the graph of a function by looking at its derivative. It is *always* true that if $f'(x) < 0$, then $f(x)$ is decreasing, and if $f'(x) > 0$, then $f(x)$ is increasing. But the case when $f'(x) = 0$ is a little trickier. In our example, we had a minimum at $x = 0$. But that is not the only thing that can happen. There are four different possible types of behavior which may occur when $f'(x) = 0$. They are illustrated below.



$f(x) = x^2$ $f(x) = -x^2$ $f(x) = x^3$ $f(x) = -x^3$

Figure 6: Possible behavior when $f'(x) = 0$.

As we saw with $f(x) = x^2$, we have a minimum when $f'(x) = 0$. But when $f(x) = -x^2$, we have a maximum when $f'(x) = 0$. In the cases when $f(x) = x^3$ and $f(x) = -x^3$, we have **inflection points** when $f'(x) = 0$.

How can we know which case occurs if we *don't* have a graph of the function? To answer this questions completely, we need to find the *second* derivative of $f(x)$. We will do this later, but for now, we need to spend a little more time studying the first derivative.

Exercises

1. Using the definition of the derivative, find $f'(x)$ if $f(x) = 3$.

2. Using the definition of the derivative, show that if $f(x) = x^3$, then $f'(x) = 3x^2$.

3. Let $f(x) = \cos(x)$.

   (a) Graph this function on desmos. Use a domain of $[0, 2\pi]$ (you can use the wrench icon in the upper right and type "pi" for π).

   (b) What is $f'\left(\frac{\pi}{2}\right)$?

   (c) Find the equation of the tangent line at $x = \frac{\pi}{2}$.

   (d) Where is $f'(x) = 0$? Remember, the domain is $[0, 2\pi]$. By inspecting the graph, decide if there is a minimum, maximum, or inflection point at these values.

   (e) Where is the function increasing?

   (f) Where is the function decreasing?

4. Let $f(x)$ be a function – you don't know exactly what $f(x)$ is, but you are given that $f'(x) = x^2(x-2)^2$. The function is defined on all real numbers.

   (a) Where is this function increasing?

   (b) Where is this function decreasing?

   (c) When is $f'(x) = 0$? Based on what you found in (a) and (b), decide if $f(x)$ has a minimum, maximum, or inflection point at these values.

   (d) You are given that $f(3) = 12$. Find an equation of the tangent line at $x = 3$.

Solutions

1.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{3-3}{h} \\
&= \lim_{h\to 0} 0 \\
&= 0
\end{aligned}$$

2.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} \left(\frac{3x^2h}{h} + \frac{3xh^2}{h} + \frac{h^3}{h} \right) \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2.
\end{aligned}$$

3. (a) Do this online.

   (b) $f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$.

   (c) We know the slope is -1 from (b). Since $f\left(\frac{\pi}{2}\right) = 0$, we use the point $\left(\frac{\pi}{2}, 0\right)$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 0 &= -1\left(x - \frac{\pi}{2}\right) \\
y &= -x + \frac{\pi}{2}
\end{aligned}$$

   (d) $f'(x) = -\sin(x) = 0$ exactly when $x = 0, \pi, 2\pi$ (given our knowledge of the unit circle and the fact that the domain is $[0, 2\pi]$). Looking at the graph, we see a local maximum at $x = 0$ and $x = 2\pi$, and a local minimum at $x = \pi$.

   (e) The function is increasing wherever we have $f'(x) > 0$. Looking at the graph of $y = -\sin(x)$ on desmos, we observe that $f'(x) = -\sin(x)$ is positive on the interval $(\pi, 2\pi)$. By visually inspecting the graph of $f(x) = \cos(x)$, we observe that $f(x)$ is increasing on this interval.

(f) The function is decreasing wherever we have $f'(x) < 0$. Looking at the graph of $y = -\sin(x)$ on desmos, we observe that $f'(x) = -\sin(x)$ is negative on the interval $(0, \pi)$. By visually inspecting the graph of $f(x) = \cos(x)$, we observe that $f(x)$ is decreasing on this interval.

4. In this problem, you are not given the graph of the function, but you should still be able to answer the following questions.

   (a) $f'(x) = x^2(x-2)^2$, but x^2 and $(x-2)^2$ are both positive. So $f'(x)$ is always positive, therefore $f(x)$ is always increasing. In interval notation, this would be $(-\infty, \infty)$.

   (b) Based on the answer to (a), $f(x)$ is never decreasing.

   (c) Since $f'(x)$ is in factored form, the zeros are easy to find: $x = 0$ and $x = 2$. Now if there were a minimum at $x = 0$, we would go from decreasing to increasing, which is impossible since $f(x)$ is never decreasing. Likewise, if there were a maximum, we would go from increasing to decreasing, again impossible. So there must be inflection points at these two values of x.

   (d) Since $f'(3) = 3^2 \cdot (3-1)^2 = 36$, the slope of the tangent line is 36. Since $f(3) = 12$, we know that $(3, 12)$ is a point on the tangent line. We can use these to get an equation of the line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 12 &= 36(x - 3) \\ y - 12 &= 36x - 108 \\ y &= 36x - 96 \end{aligned}$$